Basics of Set Theory and Functions

Lecture 04

January 16, 2024

Announcements:

This lecture is supplemented by the following readings:

- Heim & Kratzer: Ch. 1

Your first homework assignment is available and is due on January 18th.

1 Introduction

Recall that our program for this course is to develop an explanatory model of what makes it possible to understand the meaning of natural language expressions.

We formulated this program—known as **formal semantics**—as an attempt to answer the following question:

(1) **The Fundamental Question of Formal Semantics**
What is the system of rules that comprises our ability to compositionally compute the meaning of natural language expressions?

As indicated, the approach that we are taking to this question is informed and guided by the hypothesis that languages are compositional systems:

(2) **Principe of Compositionality**
The meaning of a complex expression can be computed from:
(i) the meaning of its component expressions and
(ii) their mode of combination.

A challenge we face before being able to undertake this program is the need to be more precise about what we mean when we use the vague, pre-theoretic term "meaning." We must define:

(i) the relevant notion of **meaning** that the component parts of expressions have and

(ii) the relevant notion of **meaning** that our compositional system is intended to compute.

The approach we have taken is to first consider the issue in (ii). We will then see that we can use this to guide our consideration of the issue in (i).

During our last couple of meetings, we identified the **informational content** conveyed by an utterance as the relevant concept of 'meaning' for our investigation.

(3) **Informational Content**
The information about the world that an expression conveys

We also saw the need to distinguish between (at least) three different types of informational content that can be conveyed by a sentence. These differ primarily on the basis of how they are conveyed:

(4) a. **Assertion**
Information explicitly contributed by an expression (truth conditions)

b. **Presupposition**
Information that is taken for granted to be true by an expression

c. **Implicature**
Information that is implied/inferred from an expression

Whatever our semantic rules are going to look like, our goal is to compute the kinds of meanings outlined above. As noted, we will largely focus on rules that compute the truth-conditional meaning of utterances. Along the way we will see how such a system makes sense of non-asserted content.

Before we can do anything of this, however, it will be necessary to establish some formalisms. This includes a basic background in **set theory** and **functions**.

2 Set Theory

2.1 Basic Properties of Sets

For starters, when we talk about a **set** we are essentially referring to any collection of things.

(5) **Set**
A collection of things

The most basic way to represent a set is to enclose a list of its members in curly brackets:

(6) **Representing sets**
$\{ a, b, c \}$ = the set consisting of a, b, and c

The things that make up a set are referred to as its **members** or **elements**. The following notation is to indicate (non-)membership of a set.

(7) **Set membership**

If an object a is a member/element of a set B, then we write: $a \in B$

If an object a is *not* a member/element of a set B, then we write: $a \notin B$.

(8) a. $\mathrm{a} \in \{ a, b, c \}$ b. $\mathrm{d} \notin \{ a, b, c \}$

In principle, anything can be a member of a set, including another set.

(9) **Some possible sets**

a. { ♣, ♦, ♥, ♠ } = the set consisting of card suits

b. { Jason, Kuniko, Mike, Sam } = the set consisting of the OU Linguistics professors

c. { /p/, /t/, /k/, π, a green 1995 Chevrolet Lumina, ❋ } = a difficult to describe set

d. { a, b, c, { d, e, f } } = the set consisting of a, b, c, and the set consisting of d, e, and f

e. { { f, e } } = the set consisting of the set consisting of f and e

Sets are entirely defined by the identify of their members. They are *not* defined by the order in which their members appear nor are they defined by the number of occurrences of their members.

(10) **Examples of equivalent sets**

a. $\{ a, b, c \} = \{ a, c, b \} = \{ b, a, c \} = \{ b, c, a \} = \{ c, a, b \} = \{ c, b, a \}$

b. $\{ a, a, b, b, c, c \} = \{ a, b, c \}$

c. $\{ a, a, a, a \} = \{ a \}$

Sets are formally distinct from our every day conceptions of *groups* of things.

One way in which sets are distinguished from groups is in the fact that sets can consist of only a single member. A set with a single member is referred to as a **singleton set**.

(11) **Examples of singleton sets**

a. { a } = the single set consisting of a

b. { { f, e } } = the single set consisting of the set consisting of f and e

A second way that sets can differ from groups is that sets may have no members at all. There is exactly one, unique set that doesn't have any members at all. This is the **empty set** or the **null set**.

(12) **Representing the Null/Empty Set**
The null/empty set can be written as { }. Although, it is commonly represented as $\varnothing$.

Sometimes we are interested in how many members a set has, referred to as the **cardinality** of a set.

(13) **Cardinality of a set**
The number of distinct members of a set

(14) **Representing the cardinality of set**

a. $|\{ a, b, c \}| = 3$

b. $|\{ T, F \}| = 2$

Exercise: What is the cardinality of the following sets? What are the members of each set?

(15) a. $\{ a, a, b, b \}$

b. $\{ a, \{ b \}, b, c \}$

c. $\{ 42, \varnothing \}$

d. $\{ 117, \{ 119, 120, 121 \} \}$

Are the following statements true?

(16) a. $\{ a \} = \{ a, b \}$

b. $\{ 7, 7, 9 \} = \{ 9, 7 \}$

c. Jason $\in$ { Jason, Kuniko, Mike, Sam }

d. $\{ a \} \in \{ a, \{ a \} \}$

2.2 Relations Between Sets

It is possible for sets to stand in particular relationships to other sets.

Overlap. Two sets that have one ore more members in common may be said to **overlap** in the case that A contains members not in B and vice versa.

(17) **Overlapping sets**
A set A and a set B overlap if they have at least one member in common and are not in a superset/subset relation.

(18) **Examples of overlapping sets**

a. $\{ a, b, c \}$ and $\{ c, d, e \}$ overlap.

b. $\{ w, x, y \}$ and $\{ x, y, z \}$ overlap.

Overlapping sets can be represented visually as in Figure 1:

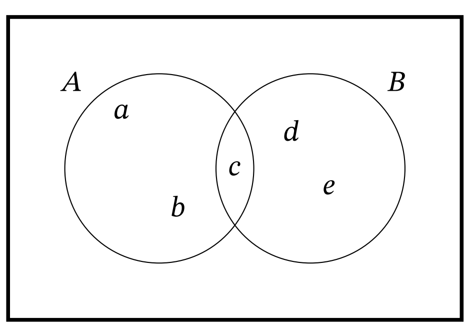

Figure 1: Sets A and B overlap.

Disjointness. If two sets do not have any members in common, they are said to be **disjoint**.

(19) **Disjoint sets**
A set A and a set B are disjoint if they have no members in common.

(20) **Examples of disjoint sets**

a. $\{ a, b, c \}$ and $\{ d, e, f \}$ are disjoint.

b. $\{ a, b, c \}$ and $\{ b, \{ c, d \} \}$ are disjoint.

Disjoint sets can be represented visually as in Figure 2:

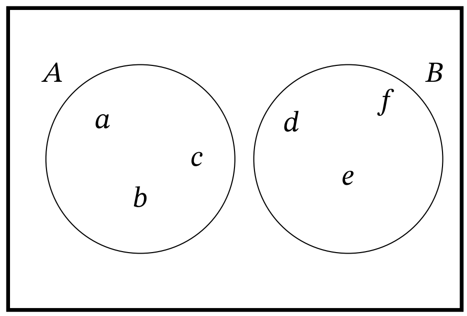

Figure 2: Sets A and B are disjoint.

Superset and Subset. If every member of a set A is also a member of a set B, then A is a **subset** of B. Likewise, B is a **superset** of A.

(21) **Subset**
A set A is a subset of B if every member of A is a member of B.

(22) **Superset**
A set A is a superset of B if every member of B is a member of A.

(23) **Examples of subsets and supersets**

a. $\{ a, b, c \}$ is a subset of $\{ a, b, c, d \}$.

b. $\{ x, y, z \}$ is a superset of $\{ x, y \}$.

(24) **Representing subset and superset relations**

If A is a subset of B, then we write $A \subseteq B$.

If B is a superset of A, then we write $B \supseteq A$.

This reveals an important constraint on sethood. Every well-formed set is a subset and a superset of itself.

(25) For every set A, $A \subseteq A$ and $A \supseteq A$

It is helpful, therefore, to ditinguish between subset/superset and **proper subsets** and **proper supersets**.

(26) **Proper Subset**
A set A is a proper subset of B if every member of A is a member of B and A is not equal to B.

(27) **Proper Superset**
A set A is a proper superset of B if every member of B is a member of A and A is not equal to B.

(28) **Representing proper subsets and proper supersets**

If $A \subseteq B$ and $A \neq B$, then we write $A \subset B$.

If $A \supseteq B$ and $A \neq B$, then we write $A \supset B$.

Proper subsets and proper supersets can be represented visually as in Figure 3:

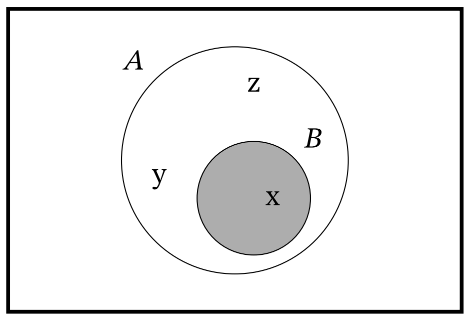

Figure 3: Set A is a proper superset of set B ($A \supset B$). Set B is a proper subset of set A ($B \subset A$).

Exercise: Are the following statements true?

(29) a. $\{ d, f \} \subseteq \{ d, e, f \}$

b. $\{ a \} \subseteq \{ a, \{ a \} \}$

c. $\{ 2, 4 \} \subset \{ 2, \{ 4 \}, 6 \}$

d. { Jason } $\nsubseteq$ { Jason, { Kuniko, Mike, Sam } }

2.3 Operations on Sets

There are several operations that can be performed on sets to define new sets.

Union. The union operation defines a new set by combining the members of two sets.

(30) **Set Union**
The union of sets A and B ($A \cup B$) is the set that contains all the members of A and all the members of B.

(31) **Examples of set union**

a. The union of $\{ a, b, c \}$ and $\{ c, d, e \}$ is $\{ a, b, c, d, e \}$

b. The union of $\{ w, x \}$ and $\{ y, z \}$ is $\{ w, x, y, z \}$

c. The union of $\{ l, m \}$ and $\{ n, o \}$ is $\{ l, m, n, o \}$

It is possible to visualize the union of two sets in the way illustrated in Figure 4.

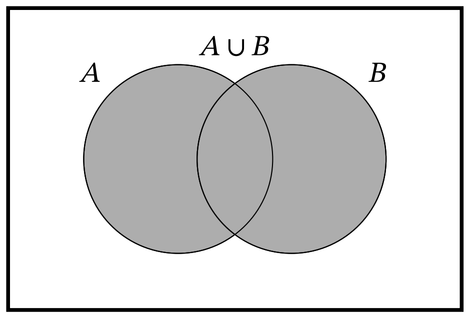

Figure 4: $A \cup B$ is the union of sets A and B.

Intersection. The intersection operation defines a new set by taking the members that two sets have in common.

(32) **Set Intersection**
The intersection of two sets A and B ($A \cap B$) is the set that contains all the elements that are both in A and in B.

(33) **Examples of set intersection**

a. The intersection of $\{ a, b, c \}$ and $\{ c, d, e \}$ is $\{ c \}$.

b. The intersection of $\{ w, x, y \}$ and $\{ x, y, z \}$ is $\{ x, y \}$

c. The intersection of $\{ l, m \}$ and $\{ n, o \}$ is $\varnothing$

It is possible to visualize the intersection of two sets in the way illustrated in Figure 5.

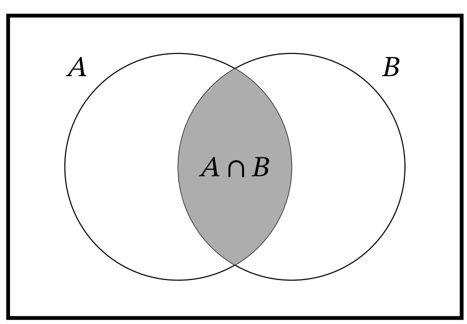

Figure 5: $A \cap B$ is the intersection of sets A and B.

Complementation. The complementation operation defines a new set by removing from one set the members that it shares with another set.

(34) **Set Complementation**
The complement set of B relative to a set A $(A - B)$ is the set that contains all the memebers in A that are not in B.

(35) **Examples of set complementation**

a. The complement of $\{ a, b, c \}$ relative to $\{ c, d, e \}$ is $\{ d, e \}$, or $\{ c, d, e \}$ minus $\{ a, b, c \}$ is $\{ d, e \}$.

b. The complement of $\{ x, y, z \}$ relative to $\{ v, w, x, y \}$ is $\{ v, w \}$, or $\{ v, w, x, y \}$ minus $\{ x, y, z \}$ is $\{ v, w \}$.

It is possible to visualize the complement of two sets in the way illustrated in Figure 6.

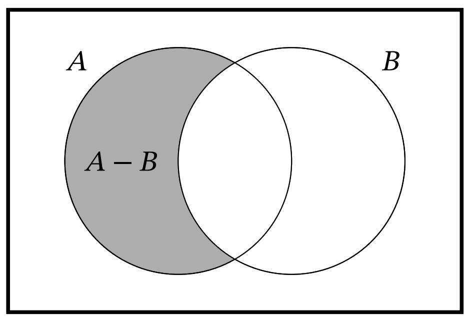

Figure 6: $A - B$ is the complement set of B relative to A.

This leads us to some noteworthy equivalences:

(36) a. If A and B overlap, then $A \cap B \neq \varnothing$

b. If A and B are disjoint, then $A \cap B = \varnothing$

Exercises: Consider the following sets:

(37)

$A = \{ a, b, c, 2, 3, 4 \}$

$B = \{ a, b \}$

$C = \{ c, 2 \}$

$D = \{ b, c \}$

$E = \{a, b, \{ c \} \}$

$F = \{ \{ a, b \}, \{ c, 2 \} \}$

List the members of each of the following sets:

(38) a. $B \cup C =$

b. $D \cap B =$

c. $C \cap F =$

d. $E - A =$

2.4 Defining Sets

Sets can have a very large, and even infinite, number of members. In these cases, it, won't be possible for us to simply list all of the members as we have done up to this point. It will therefore be convenient for us to have the ability to define and refer to sets in various ways.

Names and Variables. One way to refer to particular sets, which we have been doing, is to use names and variables. This requires that we make clear what sets those names and variables stand in for.

(39) **Referring to sets with names**

a. OU Students = the set of students at Oakland University
b. $\mathbb{N}$ = the set of all natural numbers

(40) **Referring to sets with variables**

a. F = the set of professors at Oakland University
b. C = the set of cats

Set Abstraction. Another convenient notation, which we will make significant use of, is using **set abstraction**.

(41) **Set Abstraction**

$\{ \underbrace{x}_{\textit{variable}} : \underbrace{\ldots\ x\ \ldots}_{\textit{condition}} \}$ = the set of entities x such that ... x ....

(42) **Examples of set abstraction notation**

a. $\{ x : x \text{ is a course at OU} \}$ =
the set of entities x such that x is a course at OU =
the set of courses at OU

b. $\{ x : \text{cats are allowed to eat } x \}$ =
the set of entities x such that cats are allowed to eat x =
the set of entities that cats are allowed to eat

c. $\{ x : \{ y : x \text{ hates } y \} = \varnothing \}$ =
the set of entities x such that
the set of entities y such that x hates y is null
the set of entities that don't hate anything

We can determine for any x whether it is a member of a set by plugging it in to the variable position in the set-membership condition.

(43) **Example of set membership with abstraction notation**

a. LIN 4307 $\in \{ x : x \text{ is a course at OU} \}$

b. toothpaste $\notin \{ x : \text{cats are allowed to eat } x \}$

Set abstraction notation can be used straightforwardly with notation for representing relationships between sets and operations on sets:

(44) **Example of relations and operations with abstraction notation**

a. $\{ x : x \text{ is a cat} \} \subset \{ y : y \text{ is an animal} \}$
the set of entities x such that x is a cat is
a proper subset of the set of entities y such that y is an animal

b. $\{ x : x \text{ is a LIN course at OU} \} \cup \{ y : y \text{ is a PSY course at OU} \} =$
$\{ z : z \text{ is a LIN course at OU or } z \text{ is a PSY course at OU} \}$

3 Functions

As we will see, formal semantics is built off of the idea that the meanings of certain expressions should be modeled as **functions**, which are essentially formulas for mapping between sets. Consequently, we will need a basic understanding of what functions are and how they work.

3.1 Ordered Pairs

For any two entities x and y, there is an ordered pair of those entities.

(45) **Ordered Pair**
A pair of things in an ordered sequence

(46) **Representing ordered pairs**
$\langle x, y \rangle$ is the ordered pair consisting of x followed by y.

Unlike simple sets, ordered pairs are defined in part by the order in which their elements appear. This has the following consequence:

(47) **Property of Ordered Pairs**
If $x \neq y$, then $\langle x, y \rangle \neq \langle y, x \rangle$.

Similar to sets, anything can in principle be part of an ordered pair.

(48) **Examples of some ordered pairs**

a. $\langle 12, 21 \rangle$

b. $\langle$ ♣, black $\rangle$

c. $\langle e, t \rangle$

Because anything can in principle be a member of set, it is also possible to have sets of ordered pairs.

(49) **A Set of Ordered Pairs (A Relation)**
$\{ \langle a, b \rangle, \langle a, d \rangle, \langle d, e \rangle \}$

3.2 Functions

There is a specific kind of set of ordered pairs that is crucially important to the fields of mathematics, logic, and natural language. It is so important that we've given it a name: a **function**.

(50) **Function**
A function is any set of ordered pairs f such that
for any x, if $\langle x, y\rangle \in f$ and $\langle x, z\rangle \in f$, then $y = z$.

Stated more simply, a set of ordered pairs is a function if each pair has a unique first element. Consider the following sets of ordered pairs as a point of illustration.

(51) **Examples of functions**

a. $\{ \langle 1, 1\rangle, \langle 2, 4\rangle, \langle 3, 9\rangle \}$ is a function.

b. { ⟨ Anna, orange ⟩, ⟨ Ben, yellow ⟩, ⟨ Cynthia, yellow ⟩ } is a function.

We can visualize functions as specific relations between the members of two sets in diagrams like the one in Figure 7.

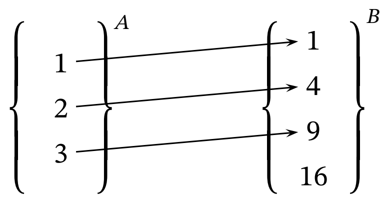

Figure 7: Representation of a function

A relation that pairs a first member with multiple different second members is, by definition, not a funciton.

(52) **Examples of non-functional relations**

a. $\{ \langle 1, 1\rangle, \langle 2, 4\rangle, \langle 2, 9\rangle \}$ is *not* a function.

b. { ⟨ Anna, orange ⟩, ⟨ Anna, yellow ⟩, ⟨ Ben, green ⟩ } is *not* a function.

We can visualize non-functional relations between the members of two sets in diagrams like the one in Figure 8.

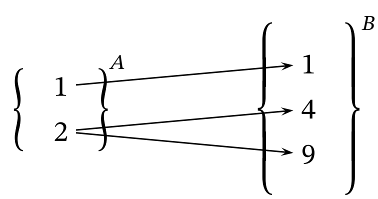

Figure 8: Representation of a non-functional relation

3.3 Some Terminology for Functions

If a set of ordered pairs f is a function, the following definitional equivalence will hold:

(53) $f(x) =_{def}$ the unique y such that $\langle x, y \rangle \in f$.

This notation is generally represented as follows:

(54) **Equation for a function**
$f(x) = y$

(55) **Terminology for the equation $\boldsymbol{f(x) = y}$**

We can refer to f as the *function*.

We can refer to x as the *argument*.

We can refer y as the *value*.

We can read $f(x) = y$ as the function f maps (the argument) x to (the value) y.

(56) **Terminology for the term $\boldsymbol{f(x)}$**

The function f applied to (the argument) x

The function f taking x as an argument

f of x

The key analogy that underlies this terminology is the idea that we can think of functions as little machines.

The ordered pairs $\langle x, y \rangle$ in the function are pairs of *inputs* and *outputs* of the machine.

The function takes the first member of the ordered pair as the *input* and spits out the second member as the *output*.

Exercise. Assume the function defined in (57).

(57) $f = \{ \langle 1, 1 \rangle, \langle 2, 4 \rangle, \langle 3, 9 \rangle \}$.

Which of the following statements are true? Why or why not?

(58) a. $f(1) = 1$

b. $f(2) = 1$

c. $f(3) = 5$

d. $f(9) = 3$

There are special terms for the sets representing the inputs and outputs of a function.

The **domain** of a function is the set of all of its possible inputs, i.e. its *arguments*.

(59) **Domain**
For any function f, the domain of f is the set of all those entities x such that there is some y such that $\langle x, y\rangle \in f$.

The **range** of a function is the set of all of its possible outputs, i.e., its *values*.

(60) **Range**
For any function f, the range of f is the set of all those entities y such that there is some y such that $\langle x, y\rangle \in f$.

As part of defining functions, it is often necessary to provide some notation to indicate the domain and range of a function.

(61) **Representing domains and ranges**
If the function f maps the set A into the set B, we can write $f : A \to B$.

We say that some x is **defined** for the function f is x is in the domain of f. If x is not in the domain of f, x is undefined for the function f.

(62) **Definedness**
Some x is defined for the function f if there is a y such that $\langle x, y\rangle \in f$.

(63) **Undefinedness**
Some x is undefined for the function f if there is no y such that $\langle x, y\rangle \in f$.

3.4 Defining Functions

We face problems when representing functions that are similar to those we faced when representing sets. Functions can be very large, and often consist of an infinite number of ordered pairs. Therefore, it will not be possible to define function simply by listing their members in set notation.

A useful alternative for defining functions involves specifying a condition that must be met by every ordered pair in the function.

(64) **Notation for functions**

$f : A \to B$
for every $x \in A$, $f(x) = \ldots\ x\ \ldots$

"The function f that maps the set A into the set B and
for every x in A, maps x to the value that meets the specified condition."

For a more concrete example, we can consider the function defined in (65).

(65) $g : \mathbb{N} \rightarrow \mathbb{N}$
for every $x \in \mathbb{N}$, $g(x) = x + 1$

"The function g that maps the set of natural numbers into the set of natural numbers and for every x, maps x to $x + 1$."

(66) $g = \{ \langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 4 \rangle, \dots \}$

For an even more concrete, concrete example, suppose that P is the set of individuals listed and C is the set of colors listed.

(67) a. $P = \{ Anna, Ben, Cynthia, Derek, Erika \}$

b. $C = \{ red, orange, yello, green, blue, purple \}$

Let there be some function $h : P \rightarrow C$ that maps people to their favorite color.

(68) $h : P \rightarrow C$
for every $x \in P$, $h(x) =$ the favorite color of x

"The function h that maps the set of people into the set of colors and for every x, maps x to the favorite color of x."

We can visualize the mapping of h as shown in Figure 9:

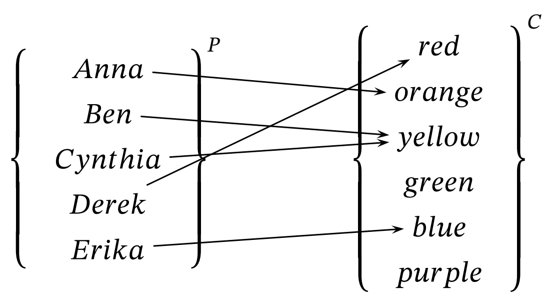

Figure 9: A diagram of the mapping of the function h

Exercise: With respect to the function above, please answer the following question.

(69) a. What is the domain of h?

b. What is the range of h?

c. Is h(Cynthia) = yellow true?

d. h(Ben) =

e. For what value x does $f(x) =$ purple?

As one more illustrative example that also provides some synthesis for where we are and where we are heading, considering the following function:

(70) $j : \mathbb{N} \to \{ \text{True}, \text{False} \}$
for every $x \in \mathbb{N}$, $j(x) = \text{True}$ *iff* x is even

"The function j that maps the set of natural numbers into the set of truth values and for every natural number x, maps x to True *iff* x is even."

(71) $g = \{ \langle 1, \text{False} \rangle, \langle 2, \text{True} \rangle, \langle 3, \text{False} \rangle, \langle 4, \text{True} \rangle, \ldots \}$

Exercise: Consider again the function in (72).

(72) $f = \{ \langle 1, 1 \rangle, \langle 2, 4 \rangle, \langle 3, 9 \rangle \}$

Provide a definition for this function using the notation for functions above.